HYPERBOLIC EQUATION OF THERMAL CONDUCTIVITY. SOLUTION OF THE DIRECT AND INVERSE PROBLEMS FOR A SEMIINFINITE BAR

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The first and second boundary-value problems as well as the linear boundary-value inverse heat-conduction problem with fixed heat collector and boundary have been solved. Account of the finite heat-propagation velocity increases the boundary-value inverse heat-conduction problem stability.

At the present time two approaches exist to solve boundary-value inverse heat-conduction problems (IHCP). The first group of methods of solving boundary-value IHCP (direct methods) use the parabolic equation of heat conductivity or its general form [1-6]. It is well known [6] that the boundary-value IHCP for the parabolic equation of heat conduction is incorrect due to its instability. This circumstance imposes restrictions on the parameters of calculation schemes used in solving the problem (approximate step in time, number of iterations, etc.), and, consequently, restricts the accuracy of the solutions obtained (particularly for fast flows and quickly varying processes). A second approach in solving the boundary-value IHCP consists of using regularized methods of its solution [5, 7, 8], in which one uses the transition from the incorrect statement of the problem to the correct one, which significantly enhances the stability of the boundary-value IHCP. It is, obviously, possible to propose many ways of this transitions. Among the variety of these methods, one must choose those in best agreement withthe real physical processes occurring in heat transfer. It is shown below that account of the finite rate of heat propagation is a natural "regularizing" factor which enhances the stability of boundary-value IHCP.

The hyperbolic heat-conduction equation [4, 9, 10]

$$\beta^2 \frac{\partial^2 u}{\partial t^2} + \frac{1}{a} \cdot \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

describes nonstationary heat-conduction process more accurately than does the parabolic equation. In this paper we derive several integral forms for solving linear inverse boundary-value problems of heat conduction in the one-dimensional case, and a method of their solution is given. At the same time, the corresponding direct boundary-value problems are solved for arbitrary conditions at the boundaries. During the process of solving IHCP, the suggested method can approximately determine the velocity of heat-propagation, and, consequently, the relaxation constant, which is an additional advantage of the method.

Statement and Solution of Direct Boundary-Value Problems for a Semiinfinite Bar. The thermal flow $q(x, t)$ is related as follows to the temperature $u(x, t)$, obeying the hyperbolic equation [4, 9, 10]:

$$\eta \frac{\partial q}{\partial t} + q = -\lambda \frac{\partial u}{\partial x}; \quad \eta = a\beta^2. \tag{2}$$

Taking into account the vanishing condition $q(x, 0) = 0$, Eq. (2) is easily integrated

$$q(x, t) = -\frac{\lambda}{\eta} \int_0^t \frac{\partial u(x, \xi)}{\partial x} \exp[-(t-\xi)/\eta]\, d\xi. \tag{2a}$$

Knowing the temperature $u(x, t)$, by Eq. (2a) one can calculate the thermal flow in any point of the body. In stating the problem of heating a semiinfinite bar ($x \in [0, \infty]$), we apply the initial homogeneous conditions

$$u(x, 0) = 0; \quad \frac{\partial u(x, 0)}{\partial t} = 0. \tag{3}$$

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At the left boundary of the bar (x = 0) we choose two types of boundary conditions

$$u(0,\ t) = u_1(t) \tag{4a}$$

or

$$q(0,\ t) = q_1(t). \tag{4b}$$

It is assumed that temperature $u_1(t)$ of the medium and the thermal flow $q_1(t)$ at the surface of the bar can be arbitrary functions of time. Using Eq. (2), we replace boundary condition (4b) by the equivalent, but more convenient

$$-\lambda \frac{\partial u(0,\ t)}{\partial x} = q_1^*(t); \quad q_1^*(t) = \eta \frac{dq_1}{dt} + q_1. \tag{4c}$$

Corresponding to the type of boundary condition at infinity, one of the following requirements must be satisfied

$$u(\infty,\ t) = 0; \quad \frac{\partial u(\infty,\ t)}{\partial x} = 0. \tag{5}$$

The problems stated are solved by the operator method under the assumption that λ, a, β are constant quantities. The Laplace transforms of $u(x,\ t)$, $q(x,\ t)$, $u_1(t)$, $q_1(t)$ and $q_1^*(t)$ are denoted by $U(x,\ p)$, $Q(x,\ p)$, $U_1(p)$, $Q_1(p)$ and $Q_1^*(p)$, respectively. In the transform region we have two solutions, corresponding to the various types of boundary conditions:

$$U = U_1(p) \exp(-x\beta \sqrt{p^2 + p/\eta}), \tag{6a}$$

$$U = Q_1^*(p) \exp(-x\beta \sqrt{p^2 + p/\eta})/\lambda\beta \sqrt{p^2 + p/\eta}. \tag{6b}$$

Returning to function $u(x,\ t)$, we obtain for it the corresponding expressions [11, 12]:

$$u = E(t - \beta x) \left\{ u_1(t - \beta x) \exp(-x/2a\beta) + \frac{\beta x}{2\eta} \int_0^{t-\beta x} u_1(\tau) \frac{I_1\left[\frac{1}{2\eta} \sqrt{(t-\tau)^2 - \beta^2 x^2}\right]}{\sqrt{(t-\tau)^2 - \beta^2 x^2}} \exp[-(t-\tau)/2\eta]\, d\tau \right\}, \tag{7a}$$

$$u = E(t - \beta x) \frac{1}{\lambda\beta} \int_0^{t-\beta x} q_1^*(\tau) I_0 \left[\frac{1}{2\eta} \sqrt{(t-\tau)^2 - \beta^2 x^2}\right] \exp[-(t-\tau)/2\eta]\, d\tau. \tag{7b}$$

In Eqs. (7) $E(\xi)$ is the function of unit jump [11], equal to 0 at $\xi < 0$, and I_0 and I_1 are first-order Bessel functions of an imaginary argument. Integration by parts of the right-hand side of Eq. (7b) with account of (4c) leads to a different form of the solution $u(x,\ t)$ for boundary condition (4b):

$$u = E(t - \beta x) \left\{ \frac{\eta}{\lambda\beta} q_1(t - \beta x) \exp(-x/2a\beta) + \right.$$

$$\left. + \frac{1}{2\lambda\beta} \int_0^{t-\beta x} q_1(\tau) \exp[-(t-\tau)/2\eta] \left[I_0\left(\frac{1}{2\eta} \sqrt{(t-\tau)^2 - \beta^2 x^2}\right) + \frac{(t-\tau)}{\sqrt{(t-\tau)^2 - \beta^2 x^2}} I_1\left(\frac{1}{2\eta} \sqrt{(t-\tau)^2 - \beta^2 x^2}\right) \right] d\tau \right\}. \tag{7c}$$

In Eq. (7c) the solution $u(x,\ t)$ is expressed in terms of the thermal flow $q_1(t)$ on the bar, and not in terms of the auxiliary function $q_1^*(t)$. Introducing the dimensionless quantities

$$\mathrm{Fo} = at/x^2, \quad \mathrm{Fo}' = a\tau/x^2, \quad \gamma = a\beta/x, \tag{8}$$

Eqs. (7a) and (7b) can be written in the form

$$u(\gamma,\ \mathrm{Fo}) = E(\mathrm{Fo} - \gamma) \left\{ u_1(\mathrm{Fo} - \gamma) \exp(-1/2\gamma) + \int_0^{\mathrm{Fo}-\gamma} K(\mathrm{Fo} - \mathrm{Fo}',\ \gamma) u_1(\mathrm{Fo}')\, d\mathrm{Fo}' \right\}, \tag{9a}$$

where

$$K(\mathrm{Fo},\ \gamma) = \frac{1}{2\gamma} \cdot \frac{I_1\left(\frac{1}{2\gamma^2} \sqrt{\mathrm{Fo}^2 - \gamma^2}\right)}{\sqrt{\mathrm{Fo}^2 - \gamma^2}} \exp(-\mathrm{Fo}/2\gamma^2), \tag{9a*}$$

$$u(\gamma, \text{Fo}) = E(\text{Fo}-\gamma)\left\{\frac{\gamma x}{\lambda} q_1(\text{Fo}-\gamma)\exp(-1/2\gamma) + \int_0^{\text{Fo}-\gamma} K(\text{Fo}-\text{Fo}', \gamma) q_1(\text{Fo}')\, d\text{Fo}'\right\}, \tag{9b}$$

where

$$K(\text{Fo}, \gamma) = \frac{x}{2\gamma\lambda}\left\{I_0\left(\frac{1}{2\gamma^2}\sqrt{\text{Fo}^2-\gamma^2}\right) + \frac{\text{Fo}}{\sqrt{\text{Fo}^2-\gamma^2}} I_1\left(\frac{1}{2\gamma^2}\sqrt{\text{Fo}^2-\gamma^2}\right)\right\}\exp(-\text{Fo}/2\gamma^2). \tag{9b*}$$

The integral relations (9a), (9b) obtained between the temperature $u(\gamma, \text{Fo})$ inside the body and the temperature $u_1(\text{Fo})$ at the boundary of the bar or the thermal flow at the surface of the bar can be used to solve various types of problems. If function $u_1(\text{Fo})$ or $q_1(\text{Fo})$ is known, Eqs. (9a), (9b), and (9a*), (9b*) are the solutions of the corresponding direct boundary-value problems.

Using the dominant terms of the asymptotic Bessel functions I_0 and I_1, it can be shown that under the condition $(\text{Fo}-\text{Fo}') > \gamma$ the limiting transition $\beta \to 0$ in Eqs. (9) leads to the corresponding solutions of the boundary-value problems for the parabolic equation of thermal conduction. If β is small, but $\beta \neq 0$, then for small Fo the solutions of (9) can differ from the corresponding solutions of the parabolic equation; with increasing Fo this difference diminishes. The problem of heating a semiinfinite bar with boundary conditions of the first kind u_1 = const was treated in [10]. The physical effects occurring in the transition from a parabolic to a hyperbolic heat-conduction equation were analyzed in similar detail (the presence of a propagating shock heat wave, restriction to a maximum heat flow and heat-transfer coefficients, etc.). Obviously, a noticeable difference in the solutions for small Fo can be observed for fast flows of intense thermal processes and processes at very low temperatures.

Solution of Linear Boundary-Value IHCP. Integral relations (9) can be used for solving inverse problems on recovering the temperature $u_1(\text{Fo})$ or the thermal flow $q_{10}\text{Fo})$ at the edge of the bar from the temperature $\psi(\text{Fo})$ measured in the internal cross section of the bar at a distance x from the edge. In this case Eqs. (9) are Volterra linear integral equations of the second kind for the unknown functions $z_\gamma(\text{Fo}) = \{u_1(\text{Fo}), q_1(\text{Fo})\}$:

$$u(\text{Fo}^* + \gamma) = \int_0^{\text{Fo}^*} K(\text{Fo}^* - \text{Fo}' + \gamma, \gamma) z_\gamma(\text{Fo}')\, d\text{Fo}' + \eta_0 z_\gamma(\text{Fo}^*)\exp(-1/2\gamma). \tag{10}$$

Here

$$\text{Fo}^* = \text{Fo} - \gamma;\ \eta_0 = 1 \text{ and } \eta_0 = x\gamma/\lambda \tag{10a}$$

for boundary conditions of the first and second kind, respectively, and the kernels $K(\text{Fo}, \gamma)$ of Eq. (10) are determined by Eqs. (9a*) and (9b*). To solve Eq. (10) one may use the well-known numerical and analytic methods [13], such as the Neumann series, the collection method, application of quadrature equations, etc.

The parameter γ in the integral equation (10) can be either given or unknown. In the latter case at some definite value γ_0 of the parameter γ, the function $z_{\gamma_0}(\text{Fo})$ will be near the real temperature or the thermal flow at the wall, therefore it can be taken as the approximate solution of the boundary-value IHCP. The quantity $1/\beta_0$ can be considered to be the approximate value of the heat-propagation velocity in the solid. The quantity β_0 is most simply determined as follows. Putting in Eq. (10) $\text{Fo}^* = 0$, we obtain a simple relation between the initial value $z_\gamma(0)$ and the measurable temperature $u(\text{Fo})$

$$z_\gamma(0) = u(\gamma)\exp(1/2\gamma)/\eta_0. \tag{11}$$

Due to the finiteness of the heat propagation velocity the function $u(\text{Fo})$ must have a discontinuity of first kind at $\text{Fo} = \gamma_0$, therefore the quantity $z_\gamma(0)$ as a function of γ also has a discontinuity of first kind at $\gamma = \gamma_0$. Figure 1 shows the curve $u(\text{Fo})$ used in the numerical experiment. The γ_0 value was taken equal to $\gamma_0 = 0.06$. In the same figure we show the γ dependence of the initial $z_\gamma(0)$. The use of the experimental curves for $u(\text{Fo})$ in constructing the function $z_\gamma(0)$ leads to the consequence that, starting with some value of γ, the quantity $z_\gamma(0)$ increases quickly from the initial vanishing value. This boundary value of γ can be approximately taken to be γ_0.

We form the function $\Phi(\gamma)$, taking into account the behavior of the solution $z_\gamma(\text{Fo})$ and its derivative $dz_\gamma(\text{Fo})/d\text{Fo}$:

$$\Phi(\gamma) = \left\| u(\text{Fo}^* + \gamma) - \int_0^{\text{Fo}^*} K(\text{Fo}^* - \text{Fo}' + \gamma, \gamma) z_\gamma(\text{Fo}')\, d\text{Fo}' - \eta_0 z_\gamma(\text{Fo}^*)\exp(1/2\gamma)\right\| + b_0\|z_\gamma(\text{Fo})\| + b_1\left\|\frac{dz_\gamma(\text{Fo})}{d\text{Fo}}\right\|. \tag{12}$$

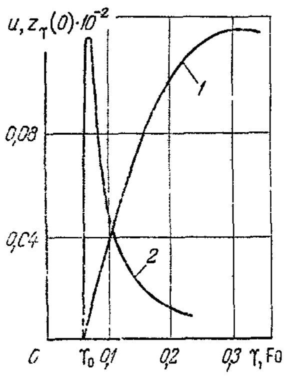

Fig. 1

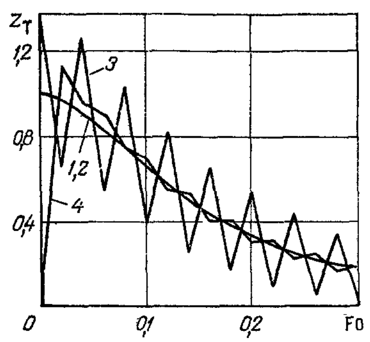

Fig. 2

Fig. 1. The relative temperature u(Fo) at a distance x from the edge of the bar and the corresponding quantity $z_\gamma(0)$: 1) the function u(Fo); 2) behavior of the initial value $z_\gamma(0)$ at the edge of the bar, constructed by Eq. (11).

Fig. 2. Dependence of the recovered relative temperature z_γ(Fo) at the edge of the bar for various values of γ: 1, 2) exact and calculated curves of z_γ (Fo); 3) oscillating curve z_γ(Fo) for $\gamma = 0.0601$; 4) z_γ(Fo) for $\gamma = 0.05$.

The norms in Eq. (12) are taken in the L_2 space, and b_0 and b_1 are weight coefficients. Since for parameter values γ near the γ_0, the solution $z_\gamma(0)$ oscillates for a suitable choice of the coefficients b_0 and b_1, a minimum of the functional Φ is achieved at a continuous solution corresponding to $\gamma = \gamma_0$. Therefore, the parameter γ_0 can be chosen by the condition

$$\Phi(\gamma_0) = \min_{\gamma} \Phi(\gamma). \tag{12a}$$

Numerical calculations recovering the temperature at the edge of the bar from function u(Fo), illustrated in Fig. 1, showed the effectiveness of choosing the parameter γ_0 from Eqs. (12), (12a). Thus, as a result of numerical calculations, the value $\gamma_0 = 0.060001$ was obtained instead of the exact value $\gamma_0 = 0.06$. In the calculations it was assumed that the weight coefficients are $b_0 = b_1 = 1$ and the functional Φ was replaced by the corresponding sum over a uniform partition of the points Fo with step H = 0.02. Integral equation (10) was solved by the method described below. Figure 2 (curves 1 and 2) shows the practically coinciding exact solution and results of calculations by Eqs. (10), (12), and (12a). The oscillating curves z_γ(Fo) for $\gamma = 0.061 > \gamma_0$ and $\gamma = 0.05 < \gamma_0$ are shown on curves 3 and 4.

A method of solving integral equations, based on approximating the solution by discontinuous step functions, is widely used in engineering practice. We apply this method to solve integral equation (10). We divide the interval $I \equiv [0, \mathrm{Fo}^*_{max}]$ into N subintervals $I_k \equiv [(k-1)H, kH]$ by the points $\mathrm{Fo}_k = kH$, where $\mathrm{Fo}_k = kH$, where $H = \mathrm{Fo}^*_{max}/N$. We solve Eq. (10), assuming that at each subinterval I_k the solution z_γ(Fo) is approximated by the function

$$z_\gamma(\mathrm{Fo}) = \{z_\gamma(kH) + z_\gamma[(k-1)H]\}/2;\ \mathrm{Fo} \in I_k. \tag{13}$$

In this way integral equation (10) reduces to a system of algebraic equations with a triangular matrix for the values of z_γ(kH)

$$\sum_{l=1}^{k} d_{k-l} z_\gamma(lH) = u(kH + \gamma) - d_k^* z_\gamma(0), \tag{14}$$

where

$$d_0 = \eta_0 \exp(-1/2\gamma) + \frac{1}{2}\int_0^H K(H + \gamma - \xi)\,d\xi, \tag{14a}$$

$$d_k^* = \frac{1}{2}\int_0^H K(kH + \gamma - \xi)\,d\xi, \quad k = 1, 2, \ldots, N, \tag{14b}$$

$$d_{k-l} = d^*_{k-l} + d^*_{k-l+1}, \quad k-l \geqslant 1, \tag{14c}$$

and the quantity $z_\gamma(0)$ is determined by Eq. (11). The values of $z_\gamma(kH)$ are conveniently calculated by using the recurrence equation:

$$z_\gamma(kH) = \frac{1}{d_0}\left\{u(kH+\gamma) - d^*_k z_\gamma(0) - \sum_{l=1}^{k-1} d_{k-l} z_\gamma(lH)\right\}. \tag{15}$$

The calculations have shown that the error in solving the integral equation (10) by the method suggested is small. It can be estimated in the usual way [14]. The spline method can be utilized to obtain a smooth solution $z_\gamma(Fo)$. We point out that the Jordan lemma is satisfied for the Laplace transforms $U_1(p)$ and $Q_1(p)$, obtained from Eq. (6) with account of expression (4c). Therefore, operation calculus methods can be used to find the original $u_1(Fo)$ and $q_1(Fo)$.

Thus, account of the finite heat propagation velocity by means of the hyperbolic equation makes it possible to solve the linear boundary-value IHCP accurately. The method suggested for solving the IHCP enables one to determine at the same time the heat propagation velocity.

NOTATION

λ	is the thermal conductivity;
a	is the thermal diffusivity;
β	is the reciprocal of heat propagation velocity;
p	is the Laplace variable;
Fo_{max}	is the maximum value of Fo.

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